Spectral Theory

1 Eigenvalues and Eigenvectors of a Matrix

1.1. Definition of Eigenvectors and Eigenvalues

Definition 7.2: Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix and let $X \in \mathbb{C}^n$ be a ***nonzero vector*** *for which*

$$AX = \lambda X \tag{7.1}$$

for some scalar λ. Then λ is called an ***eigenvalue*** *of the matrix A and X is called an* ***eigenvector*** *of A associated with λ, or a λ-eigenvector of A.*
The set of all eigenvalues of an $n \times n$ matrix A is denoted by $\sigma(A)$ and is referred to as the ***spectrum*** *of A.*

Let's look at eigenvectors in more detail. Suppose X satisfies 7.1. Then

$$\begin{gathered} AX - \lambda X = 0 \\ \text{or} \\ (A - \lambda I)X = 0 \end{gathered}$$

for some $X \neq 0$. Equivalently you could write $(\lambda I - A)X = 0$, which is more commonly used. Hence, when we are looking for eigenvectors, we are looking for nontrivial solutions to this homogeneous system of equations!

Recall that the solutions to a homogeneous system of equations consist of basic solutions, and the linear combinations of those basic solutions. In this context, we call the basic solutions of the equation $(\lambda I - A)X = 0$ **basic eigenvectors**. It follows that any (nonzero) linear combination of basic eigenvectors is again an eigenvector.

Suppose the matrix $(\lambda I - A)$ is invertible, so that $(\lambda I - A)^{-1}$ exists. Then the following equation would be true.

$$\begin{aligned} X &= IX \\ &= \left((\lambda I - A)^{-1}(\lambda I - A)\right)X \\ &= (\lambda I - A)^{-1}((\lambda I - A)X) \\ &= (\lambda I - A)^{-1}0 \\ &= 0 \end{aligned}$$

This claims that $X = 0$. However, we have required that $X \neq 0$. Therefore $(\lambda I - A)$ cannot have an inverse!

Recall that if a matrix is not invertible, then its determinant is equal to 0. Therefore we can conclude that

$$\det(\lambda I - A) = 0 \tag{7.2}$$

Note that this is equivalent to $\det(A - \lambda I) = 0$.

The expression $\det(xI - A)$ is a polynomial (in the variable x) called the **characteristic polynomial of** A, and $\det(xI - A) = 0$ is called the **characteristic equation**. For this reason we may also refer to the eigenvalues of A as **characteristic values**, but the former is often used for historical reasons.

Theorem 7.3: The Existence of an Eigenvector

Let A be an $n \times n$ matrix and suppose $\det(\lambda I - A) = 0$ for some $\lambda \in \mathbb{C}$.
Then λ is an eigenvalue of A and thus there exists a nonzero vector $X \in \mathbb{C}^n$ such that $AX = \lambda X$.

1.2. Finding Eigenvectors and Eigenvalues

Definition 7.4: Multiplicity of an Eigenvalue

Let A be an $n \times n$ matrix with characteristic polynomial given by $\det(xI - A)$. Then, the multiplicity of an eigenvalue λ of A is the number of times λ occurs as a root of that characteristic polynomial.

For example, suppose the characteristic polynomial of A is given by $(x-2)^2$. Solving for the roots of this polynomial, we set $(x-2)^2 = 0$ and solve for x. We find that $\lambda = 2$ is a root that occurs twice. Hence, in this case, $\lambda = 2$ is an eigenvalue of A of multiplicity equal to 2.

Procedure 7.5: Finding Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. *First, find the eigenvalues λ of A by solving the equation $\det(xI - A) = 0$.*
2. *For each λ, find the basic eigenvectors $X \neq 0$ by finding the basic solutions to $(\lambda I - A)X = 0$.*

To verify your work, make sure that $AX = \lambda X$ for each λ and associated eigenvector X.

Example 7.6: Find the Eigenvalues and Eigenvectors

Let $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$. Find its eigenvalues and eigenvectors.

Solution. We will use Procedure 7.5. First we find the eigenvalues of A by solving the equation

$$\det(xI - A) = 0$$

This gives

$$\begin{aligned} \det\left(x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} \right) &= 0 \\ \det \begin{bmatrix} x+5 & -2 \\ 7 & x-4 \end{bmatrix} &= 0 \end{aligned}$$

Computing the determinant as usual, the result is

$$x^2 + x - 6 = 0$$

Solving this equation, we find that $\lambda_1 = 2$ and $\lambda_2 = -3$.

Now we need to find the basic eigenvectors for each λ. First we will find the eigenvectors for $\lambda_1 = 2$. We wish to find all vectors $X \neq 0$ such that $AX = 2X$. These are the solutions to $(2I - A)X = 0$.

$$\begin{aligned}\left(2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 7 & -2 \\ 7 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned}$$

The augmented matrix for this system and corresponding reduced row-echelon form are given by

$$\left[\begin{array}{cc|c} 7 & -2 & 0 \\ 7 & -2 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 1 & -\frac{2}{7} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The solution is any vector of the form

$$\begin{bmatrix} \frac{2}{7}s \\ s \end{bmatrix} = s\begin{bmatrix} \frac{2}{7} \\ 1 \end{bmatrix}$$

Multiplying this vector by 7 we obtain a simpler description for the solution to this system, given by

$$t\begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

This gives the basic eigenvector for $\lambda_1 = 2$ as

$$\begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

To check, we verify that $AX = 2X$ for this basic eigenvector.

$$\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

This is what we wanted, so we know this basic eigenvector is correct.

Next we will repeat this process to find the basic eigenvector for $\lambda_2 = -3$. We wish to find all vectors $X \neq 0$ such that $AX = -3X$. These are the solutions to $((-3)I - A)X = 0$.

$$\begin{aligned}\left((-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & -2 \\ 7 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned}$$

The augmented matrix for this system and corresponding reduced row-echelon form are given by

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 7 & -7 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The solution is any vector of the form

$$\begin{bmatrix} s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This gives the basic eigenvector for $\lambda_2 = -3$ as

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To check, we verify that $AX = -3X$ for this basic eigenvector.

$$\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This is what we wanted, so we know this basic eigenvector is correct. ♠

Example 7.7: Find the Eigenvalues and Eigenvectors

Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix}$$

Solution. We will use Procedure 7.5. First we need to find the eigenvalues of A. Recall that they are the solutions of the equation

$$\det(xI - A) = 0$$

In this case the equation is

$$\det \left(x \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix} \right) = 0$$

which becomes

$$\det \begin{bmatrix} x-5 & 10 & 5 \\ -2 & x-14 & -2 \\ 4 & 8 & x-6 \end{bmatrix} = 0$$

Using Laplace Expansion, compute this determinant and simplify. The result is the following equation.

$$(x-5)\left(x^2 - 20x + 100\right) = 0$$

Solving this equation, we find that the eigenvalues are $\lambda_1 = 5, \lambda_2 = 10$ and $\lambda_3 = 10$. Notice that 10 is a root of multiplicity two due to

$$x^2 - 20x + 100 = (x-10)^2$$

Therefore, $\lambda_2 = 10$ is an eigenvalue of multiplicity two.

Now that we have found the eigenvalues for A, we can compute the eigenvectors.

First we will find the basic eigenvectors for $\lambda_1 = 5$. In other words, we want to find all non-zero vectors X so that $AX = 5X$. This requires that we solve the equation $(5I - A)X = 0$ for X as follows.

$$\left(5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

That is you need to find the solution to

$$\begin{bmatrix} 0 & 10 & 5 \\ -2 & -9 & -2 \\ 4 & 8 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By now this is a familiar problem. You set up the augmented matrix and row reduce to get the solution. Thus the matrix you must row reduce is

$$\left[\begin{array}{rrr|r} 0 & 10 & 5 & 0 \\ -2 & -9 & -2 & 0 \\ 4 & 8 & -1 & 0 \end{array}\right]$$

The reduced row-echelon form is

$$\left[\begin{array}{rrr|r} 1 & 0 & -\frac{5}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so the solution is any vector of the form

$$\begin{bmatrix} \frac{5}{4}s \\ -\frac{1}{2}s \\ s \end{bmatrix} = s \begin{bmatrix} \frac{5}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

where $s \in \mathbb{R}$. If we multiply this vector by 4, we obtain a simpler description for the solution to this system, as given by

$$t \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} \tag{7.3}$$

where $t \in \mathbb{R}$. Here, the basic eigenvector is given by

$$X_1 = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$$

It is a good idea to check your work! To do so, we will take the original matrix and multiply by the basic eigenvector X_1. We check to see if we get $5X_1$.

$$\begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 25 \\ -10 \\ 20 \end{bmatrix} = 5 \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$$

This is what we wanted, so we know that our calculations were correct.

Next we will find the basic eigenvectors for $\lambda_2, \lambda_3 = 10$. These vectors are the basic solutions to the equation,

$$\left(10 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

That is you must find the solutions to

$$\left[\begin{array}{rrr} 5 & 10 & 5 \\ -2 & -4 & -2 \\ 4 & 8 & 4 \end{array}\right]\left[\begin{array}{c} x \\ y \\ z \end{array}\right]=\left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

Consider the augmented matrix

$$\left[\begin{array}{rrr|r} 5 & 10 & 5 & 0 \\ -2 & -4 & -2 & 0 \\ 4 & 8 & 4 & 0 \end{array}\right]$$

The reduced row-echelon form for this matrix is

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so the eigenvectors are of the form

$$\left[\begin{array}{c} -2s-t \\ s \\ t \end{array}\right]=s\left[\begin{array}{r} -2 \\ 1 \\ 0 \end{array}\right]+t\left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array}\right]$$

Note that you can't pick t and s both equal to zero because this would result in the zero vector and eigenvectors are never equal to zero.

Here, there are two basic eigenvectors, given by

$$X_2=\left[\begin{array}{r} -2 \\ 1 \\ 0 \end{array}\right], X_3=\left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array}\right]$$

Taking any (nonzero) linear combination of X_2 and X_3 will also result in an eigenvector for the eigenvalue $\lambda = 10$. As in the case for $\lambda = 5$, always check your work! For the first basic eigenvector, we can check $AX_2 = 10X_2$ as follows.

$$\left[\begin{array}{rrr} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{array}\right]\left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array}\right]=\left[\begin{array}{r} -10 \\ 0 \\ 10 \end{array}\right]=10\left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array}\right]$$

Example 7.8: A Zero Eigenvalue

Let

$$A=\left[\begin{array}{rrr} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{array}\right]$$

Find the eigenvalues and eigenvectors of A.

Solution. First we find the eigenvalues of A. We will do so using Definition 7.2.

In order to find the eigenvalues of A, we solve the following equation.

$$\det(xI-A)=\det\left[\begin{array}{ccc} x-2 & -2 & 2 \\ -1 & x-3 & 1 \\ 1 & -1 & x-1 \end{array}\right]=0$$

This reduces to $x^3 - 6x^2 + 8x = 0$. You can verify that the solutions are $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 4$. Notice that while eigenvectors can never equal 0, it is possible to have an eigenvalue equal to 0.

Now we will find the basic eigenvectors. For $\lambda_1 = 0$, we need to solve the equation $(0I - A)X = 0$. This equation becomes $-AX = 0$, and so the augmented matrix for finding the solutions is given by

$$\left[\begin{array}{rrr|r} -2 & -2 & 2 & 0 \\ -1 & -3 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

The reduced row-echelon form is

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, the eigenvectors are of the form $t \left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array}\right]$ where $t \neq 0$ and the basic eigenvector is given by

$$X_1 = \left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array}\right]$$

We can verify that this eigenvector is correct by checking that the equation $AX_1 = 0X_1$ holds. The product AX_1 is given by

$$AX_1 = \left[\begin{array}{rrr} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{array}\right] \left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array}\right] = \left[\begin{array}{r} 0 \\ 0 \\ 0 \end{array}\right]$$

This clearly equals $0X_1$, so the equation holds. Hence, $AX_1 = 0X_1$ and so 0 is an eigenvalue of A.

1.3. Eigenvalues and Eigenvectors for Special Types of Matrices

Definition 7.9: Similar Matrices

Let A and B be $n \times n$ matrices. Suppose there exists an invertible matrix P such that

$$A = P^{-1}BP$$

*Then A and B are called **similar matrices**.*

Lemma 7.10: Similar Matrices and Eigenvalues

Let A and B be similar matrices, so that $A = P^{-1}BP$ where A, B are $n \times n$ matrices and P is invertible. Then A, B have the same eigenvalues.

Example 7.11: Simplify Using Elementary Matrices

Find the eigenvalues for the matrix

$$A = \left[\begin{array}{rrr} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{array}\right]$$

Solution. This matrix has big numbers and therefore we would like to simplify as much as possible before computing the eigenvalues.

We will do so using row operations. First, add 2 times the second row to the third row. To do so, left multiply A by $E(2,2)$. Then right multiply A by the inverse of $E(2,2)$ as illustrated.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix}$$

By Lemma 7.10, the resulting matrix has the same eigenvalues as A where here, the matrix $E(2,2)$ plays the role of P.

We do this step again, as follows. In this step, we use the elementary matrix obtained by adding -3 times the second row to the first row.

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{bmatrix} \tag{7.4}$$

Again by Lemma 7.10, this resulting matrix has the same eigenvalues as A. At this point, we can easily find the eigenvalues. Let

$$B = \begin{bmatrix} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{bmatrix}$$

Then, we find the eigenvalues of B (and therefore of A) by solving the equation $\det(xI - B) = 0$. You should verify that this equation becomes

$$(x+2)(x+2)(x-3) = 0$$

Solving this equation results in eigenvalues of $\lambda_1 = -2, \lambda_2 = -2$, and $\lambda_3 = 3$. Therefore, these are also the eigenvalues of A.

♠

Example 7.12: Eigenvalues for a Triangular Matrix

Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{bmatrix}$. *Find the eigenvalues of* A.

Solution. We need to solve the equation $\det(xI - A) = 0$ as follows

$$\det(xI - A) = \det \begin{bmatrix} x-1 & -2 & -4 \\ 0 & x-4 & -7 \\ 0 & 0 & x-6 \end{bmatrix} = (x-1)(x-4)(x-6) = 0$$

Solving the equation $(x-1)(x-4)(x-6) = 0$ for x results in the eigenvalues $\lambda_1 = 1, \lambda_2 = 4$ and $\lambda_3 = 6$. Thus the eigenvalues are the entries on the main diagonal of the original matrix. ♠

2 Diagonalization

2.1. Similarity and Diagonalization

Recall that if A, B are two $n \times n$ matrices, then they are **similar** if and only if there exists an invertible matrix P such that

$$A = P^{-1}BP$$

Definition 7.14: Trace of a Matrix

If $A = [a_{ij}]$ is an $n \times n$ matrix, then the trace of A is

$$\text{trace}(A) = \sum_{i=1}^{n} a_{ii}.$$

Lemma 7.15: Properties of Trace

For $n \times n$ matrices A and B, and any $k \in \mathbb{R}$,

1. $\text{trace}(A+B) = \text{trace}(A) + \text{trace}(B)$
2. $\text{trace}(kA) = k \cdot \text{trace}(A)$
3. $\text{trace}(AB) = \text{trace}(BA)$

The following theorem includes a reference to the characteristic polynomial of a matrix. Recall that for any $n \times n$ matrix A, the characteristic polynomial of A is $c_A(x) = \det(xI - A)$.

Theorem 7.16: Properties of Similar Matrices

If A and B are $n \times n$ matrices and $A \sim B$, then

1. $\det(A) = \det(B)$
2. $\text{rank}(A) = \text{rank}(B)$
3. $\text{trace}(A) = \text{trace}(B)$
4. $c_A(x) = c_B(x)$
5. *A and B have the same eigenvalues*

Definition 7.17: Diagonalizable

Let A be an $n \times n$ matrix. Then A is said to be ***diagonalizable*** *if there exists an invertible matrix P such that*

$$P^{-1}AP = D$$

where D is a diagonal matrix.

2.2. Diagonalizing a Matrix

Theorem 7.18: Eigenvectors and Diagonalizable Matrices

An $n \times n$ *matrix* A *is diagonalizable if and only if there is an invertible matrix* P *given by*

$$P = \left[\begin{array}{cccc} X_1 & X_2 & \cdots & X_n \end{array} \right]$$

where the X_k *are eigenvectors of* A.
Moreover if A *is diagonalizable, the corresponding eigenvalues of* A *are the diagonal entries of the diagonal matrix* D.

Example 7.19: Diagonalize a Matrix

Let

$$A = \left[\begin{array}{rrr} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{array} \right]$$

Find an invertible matrix P *and a diagonal matrix* D *such that* $P^{-1}AP = D$.

Solution. By Theorem 7.18 we use the eigenvectors of A as the columns of P, and the corresponding eigenvalues of A as the diagonal entries of D.

First, we will find the eigenvalues of A. To do so, we solve $\det(xI - A) = 0$ as follows.

$$\det \left(x \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] - \left[\begin{array}{rrr} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{array} \right] \right) = 0$$

This computation is left as an exercise, and you should verify that the eigenvalues are $\lambda_1 = 2, \lambda_2 = 2$, and $\lambda_3 = 6$.

Next, we need to find the eigenvectors. We first find the eigenvectors for $\lambda_1, \lambda_2 = 2$. Solving $(2I - A)X = 0$ to find the eigenvectors, we find that the eigenvectors are

$$t \left[\begin{array}{r} -2 \\ 1 \\ 0 \end{array} \right] + s \left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array} \right]$$

where t, s are scalars. Hence there are two basic eigenvectors which are given by

$$X_1 = \left[\begin{array}{r} -2 \\ 1 \\ 0 \end{array} \right], X_2 = \left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array} \right]$$

You can verify that the basic eigenvector for $\lambda_3 = 6$ is $X_3 = \left[\begin{array}{r} 0 \\ 1 \\ -2 \end{array} \right]$

Then, we construct the matrix P as follows.

$$P = \left[\begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \right] = \left[\begin{array}{rrr} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{array} \right]$$

That is, the columns of P are the basic eigenvectors of A. Then, you can verify that

$$P^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

Thus,

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \end{aligned}$$

You can see that the result here is a diagonal matrix where the entries on the main diagonal are the eigenvalues of A. We expected this based on Theorem 7.18. Notice that eigenvalues on the main diagonal *must* be in the same order as the corresponding eigenvectors in P. ♠

Theorem 7.20: Linearly Independent Eigenvectors

Let A be an $n \times n$ matrix, and suppose that A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. For each i, let X_i be a λ_i-eigenvector of A. Then $\{X_1, X_2, \ldots, X_m\}$ is linearly independent.

Corollary 7.21: Distinct Eigenvalues

Let A be an $n \times n$ matrix and suppose it has n distinct eigenvalues. Then it follows that A is diagonalizable.

Example 7.22: A Matrix which cannot be Diagonalized

Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

If possible, find an invertible matrix P and diagonal matrix D so that $P^{-1}AP = D$.

Solution. Through the usual procedure, we find that the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 1$. To find the eigenvectors, we solve the equation $(\lambda I - A)X = 0$. The matrix $(\lambda I - A)$ is given by

$$\begin{bmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{bmatrix}$$

Substituting in $\lambda = 1$, we have the matrix

$$\begin{bmatrix} 1-1 & -1 \\ 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Then, solving the equation $(\lambda I - A)X = 0$ involves carrying the following augmented matrix to its reduced row-echelon form.

$$\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Then the eigenvectors are of the form

$$t\left[\begin{array}{c}1\\0\end{array}\right]$$

and the basic eigenvector is

$$X_1=\left[\begin{array}{c}1\\0\end{array}\right]$$

In this case, the matrix A has one eigenvalue of multiplicity two, but only one basic eigenvector. In order to diagonalize A, we need to construct an invertible 2×2 matrix P. However, because A only has one basic eigenvector, we cannot construct this P. Notice that if we were to use X_1 as both columns of P, P would not be invertible. For this reason, we cannot repeat eigenvectors in P.

Hence this matrix cannot be diagonalized. ♠

Definition 7.23: Eigenspace

Let A be an $n\times n$ matrix and $\lambda\in\mathbb{R}$. The eigenspace of A corresponding to λ, written $E_\lambda(A)$ is the set of all eigenvectors corresponding to λ.

Lemma 7.24: Dimension of the Eigenspace

If A is an $n\times n$ matrix, then

$$\dim(E_\lambda(A))\leq m$$

where λ is an eigenvalue of A of multiplicity m.

Theorem 7.25: Diagonalizability Condition

Let A be an $n\times n$ matrix A. Then A is diagonalizable if and only if for each eigenvalue λ of A, $\dim(E_\lambda(A))$ is equal to the multiplicity of λ.

.3 Applications of Spectral Theory

3.1. Raising a Matrix to a High Power

Suppose A is diagonalizable, so that $P^{-1}AP=D$. We can rearrange this equation to write $A=PDP^{-1}$.

Now, consider A^2. Since $A=PDP^{-1}$, it follows that

$$A^2=\left(PDP^{-1}\right)^2=PDP^{-1}PDP^{-1}=PD^2P^{-1}$$

Similarly,

$$A^3=\left(PDP^{-1}\right)^3=PDP^{-1}PDP^{-1}PDP^{-1}=PD^3P^{-1}$$

In general,

$$A^n=\left(PDP^{-1}\right)^n=PD^nP^{-1}$$

Therefore, we have reduced the problem to finding D^n. In order to compute D^n, then because D is diagonal we only need to raise every entry on the main diagonal of D to the power of n.

Example 7.27: Raising a Matrix to a High Power

Let $A = \left[\begin{array}{rrr} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{array} \right]$. *Find* A^{50}.

Solution. We will first diagonalize A. The steps are left as an exercise and you may wish to verify that the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 1$, and $\lambda_3 = 2$.

The basic eigenvectors corresponding to $\lambda_1, \lambda_2 = 1$ are

$$X_1 = \left[\begin{array}{r} 0 \\ 0 \\ 1 \end{array} \right], X_2 = \left[\begin{array}{r} -1 \\ 1 \\ 0 \end{array} \right]$$

The basic eigenvector corresponding to $\lambda_3 = 2$ is

$$X_3 = \left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array} \right]$$

Now we construct P by using the basic eigenvectors of A as the columns of P. Thus

$$P = \left[\begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \right] = \left[\begin{array}{rrr} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right]$$

Then also

$$P^{-1} = \left[\begin{array}{rrr} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{array} \right]$$

which you may wish to verify.

Then,

$$\begin{aligned} P^{-1}AP &= \left[\begin{array}{rrr} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{array} \right] \left[\begin{array}{rrr} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{array} \right] \left[\begin{array}{rrr} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right] \\ &= \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{array} \right] \\ &= D \end{aligned}$$

Now it follows by rearranging the equation that

$$A = PDP^{-1} = \left[\begin{array}{rrr} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right] \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{array} \right] \left[\begin{array}{rrr} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{array} \right]$$

Therefore,

$$\begin{aligned} A^{50} &= PD^{50}P^{-1} \\ &= \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{50} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \end{aligned}$$

By our discussion above, D^{50} is found as follows.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{50} = \begin{bmatrix} 1^{50} & 0 & 0 \\ 0 & 1^{50} & 0 \\ 0 & 0 & 2^{50} \end{bmatrix}$$

It follows that

$$\begin{aligned} A^{50} &= \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1^{50} & 0 & 0 \\ 0 & 1^{50} & 0 \\ 0 & 0 & 2^{50} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2^{50} & -1+2^{50} & 0 \\ 0 & 1 & 0 \\ 1-2^{50} & 1-2^{50} & 1 \end{bmatrix} \end{aligned}$$

Theorem 7.28: Principal Axis Theorem

The following conditions are equivalent for an $n \times n$ matrix A:

1. *A is symmetric.*
2. *A has an orthonormal set of eigenvectors.*
3. *A is orthogonally diagonalizable.*

Example 7.29: Orthogonal Diagonalization of a Symmetric Matrix

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$. *Find an orthogonal matrix P such that $P^T AP$ is a diagonal matrix.*

Solution. In this case, verify that the eigenvalues are 2 and 1. First we will find an eigenvector for the eigenvalue 2. This involves row reducing the following augmented matrix.

$$\left[\begin{array}{ccc|c} 2-1 & 0 & 0 & 0 \\ 0 & 2-\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 2-\frac{3}{2} & 0 \end{array}\right]$$

The reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so an eigenvector is

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Finally to obtain an eigenvector of length one (unit eigenvector) we simply divide this vector by its length to yield:

$$\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Next consider the case of the eigenvalue 1. To obtain basic eigenvectors, the matrix which needs to be row reduced in this case is

$$\left[\begin{array}{ccc|c} 1-1 & 0 & 0 & 0 \\ 0 & 1-\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1-\frac{3}{2} & 0 \end{array}\right]$$

The reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, the eigenvectors are of the form

$$\begin{bmatrix} s \\ -t \\ t \end{bmatrix}$$

Note that all these vectors are automatically orthogonal to eigenvectors corresponding to the first eigenvalue. This follows from the fact that A is symmetric, as mentioned earlier.

We obtain basic eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Since they are themselves orthogonal (by luck here) we do not need to use the Gram-Schmidt process and instead simply normalize these vectors to obtain

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

An orthogonal matrix P to orthogonally diagonalize A is then obtained by letting these basic vectors be the columns.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

We verify this works. P^TAP is of the form

$$\begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

which is the desired diagonal matrix. ♠